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INHOMOGENEOUS LAYER BONDED TO A HALF-SPACE UNDER THE ACTION OF INTERNAL AND EXTERNAL FORCES

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We obtain a solution of the problem on the stress-strain state of an inhomogeneous isotropic layer, the elastic characteristics of which are bounded and integrable functions of a single Cartesian coordinate. The layer is bonded continuously to a homogeneous half-space, and is acted upon by the mass forces.

The problem arises in the analysis of coverings. The earlier papers dealt with particular cases in which an open surface was acted upon by a normal load. In [1, 2] such a problem was studied for an exponential law of variation of the modulus of elasticity with depth, with the Poisson's ratio remaining constant, while in [3] the same problem was studied for a hyperbolic law. This was done by considering an axisymmetric deformation of an inhomogeneous layer resting on a perfectly rigid support. In [4] a solution was obtained for an incompressible material in which the shear modulus changed linearly with depth, while in [5, 6] a solution was obtained for an arbitrary law of change and a constant Poisson's ratio. The method used in the latter case deserves attention. In the course of solution the layer was replaced by a system of n interconnected homogeneous isotropic plates of equal thickness, the elasticity moduli of which were defined by a given function of the inhomogeneity. Passage to the limit $n \to \infty$ gave a formally exact solution of the initial problem. It appears, that the action of shearing loads and forces applied within the layer has, so far, not been investigated.

1. Using the method developed in [7], we split the system of equilibrium equations

written in displacements for an inhomogeneous isotropic medium, the shear modulus G and the Poisson's ratio ν of which are both functions of the Cartesian coordinate z, into two subsystems

$$\frac{2G}{1-2\nu}\left[(1-\nu)\Delta S_1+\nu\frac{\partial u_z}{\partial z}\right]+\frac{\partial}{\partial z}\left[G\left(\frac{\partial S_1}{\partial z}+u_z\right)\right]+\varphi_1=0 \quad (1.1)$$

$$\Delta G\left(\frac{\partial S_1}{\partial z}+u_z\right)+\frac{\partial}{\partial z}\left\{\frac{2G}{1-2\nu}\left[\nu\Delta S_1+(1-\nu)\frac{\partial u_z}{\partial z}\right]\right\}+Z=0$$

$$\left(G\nabla^2+\frac{dG}{dz}\frac{\partial}{\partial z}\right)N_1+\varphi_2=0 \quad (1.2)$$

$$X=\frac{\partial\varphi_1}{\partial x}+\frac{\partial\varphi_2}{\partial y},\quad Y=\frac{\partial\varphi_1}{\partial y}-\frac{\partial\varphi_2}{\partial x},\quad \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

Here X, Y and Z are mass forces per unit volume.

The components u_x and u_y of the displacement vector can be written in terms of the functions S_1 and N_1 as follows:

$$u_x=\frac{\partial S_1}{\partial x}+\frac{\partial N_1}{\partial y},\qquad u_y=\frac{\partial S_1}{\partial y}-\frac{\partial N_1}{\partial x}$$

Setting in (1.1) and (1.2)

$$u_z=S_2,\quad \frac{2G}{1-2\nu}\left[\nu\Delta S_1+(1-\nu)\frac{\partial u_z}{\partial z}\right]=S_3$$

$$G\left(\frac{\partial S_1}{\partial z}+u_z\right)=S_4,\qquad G\frac{\partial N_1}{\partial z}=N_2$$

we obtain the following systems:

$$\frac{\partial S_1}{\partial z}=-S_2+\frac{1}{G}S_4,\quad \frac{dS_2}{dz}=-\frac{\nu}{1-\nu}\Delta S_1+\frac{1-2\nu}{2(1-\nu)G}S_3 \quad (1.3)$$

$$\frac{dS_3}{dz}=-\Delta S_4-Z,\quad \frac{dS_4}{dz}=-\frac{2G}{1-\nu}\Delta S_1-\frac{\nu}{1-\nu}S_3-\varphi_1$$

$$\frac{\partial N_1}{\partial z}=\frac{1}{G}N_2,\qquad \frac{\partial N_2}{\partial z}=-G\Delta N_1-\varphi_2 \quad (1.4)$$

Thus the solution of any problem of the theory of elasticity can be reduced to obtaining the solutions of the systems (1.3) and (1.4) satisfying the given boundary conditions, for the region occupied by the body.

All components of the stress tensor can be expressed in terms of the unknown functions introduced above and of their derivatives in x and y only:

$$\sigma_x=2G\left(\frac{\nu}{1-\nu}\Delta+\frac{\partial^2}{\partial x^2}\right)S_1+\frac{\nu}{1-\nu}S_3+2G\frac{\partial^2 N_1}{\partial x\partial y} \quad (1.5)$$

$$\sigma_y=2G\left(\frac{\nu}{1-\nu}\Delta+\frac{\partial^2}{\partial y^2}\right)S_1+\frac{\nu}{1-\nu}S_3-2G\frac{\partial^2 N_1}{\partial x\partial y}$$

$$\sigma_z=S_3,\quad \tau_{xy}=2G\frac{\partial^2 S_1}{\partial x\partial y}-G\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)N_1$$

$$\tau_{xz}=\frac{\partial S_4}{\partial x}+\frac{\partial N_2}{\partial y},\qquad \tau_{yz}=\frac{\partial S_4}{\partial y}-\frac{\partial N_2}{\partial x}$$

In the cylindrical coordinates r, β, z the components of the displacement vector and stress tensor are written in the form

$$u_r = \frac{\partial S_1}{\partial r} + \frac{1}{r}\frac{\partial N_1}{\partial \beta}, \quad u_\beta = \frac{1}{r}\frac{\partial S_1}{\partial \beta} - \frac{\partial N_1}{\partial r}, \quad u_z = S_2 \tag{1.6}$$

$$\sigma_z = S_3, \quad \sigma_r = 2G\left(\frac{\nu}{1-\nu}\Delta + \frac{\partial^2}{\partial r^2}\right)S_1 + \frac{\nu}{1-\nu}S_3 + 2G\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial N_1}{\partial \beta}\right)$$

$$\sigma_\beta = 2G\left(\frac{\nu}{1-\nu}\Delta + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \beta^2}\right)S_1 + \frac{\nu}{1-\nu}S_3 - \frac{2G}{r}\frac{\partial}{\partial \beta}\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)N_1$$

$$\tau_{r\beta} = \frac{2G}{r}\frac{\partial}{\partial \beta}\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)S_1 - G\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \beta^2}\right)N_1$$

$$\tau_{rz} = \frac{\partial S_4}{\partial r} + \frac{1}{r}\frac{\partial N_2}{\partial \beta}, \qquad \tau_{\beta z} = \frac{1}{r}\frac{\partial S_4}{\partial \beta} - \frac{\partial N_2}{\partial r}$$

The results obtained allow us to solve a number of novel problems in the theory of elasticity on the inhomogeneous media. We consider two of these problems.

2. An inhomogeneous layer $0 \leqslant z \leqslant h$ bonded continuously to a homogeneous isotropic half-space is acted upon by the mass forces $Z = Z(r, \beta, z)$ (Fig. 1). The variation of the elastic properties of the layer with depth is described by the functions $G^{(1)} = G^{(1)}(z)$ and $\nu^{(1)} = \nu^{(1)}(z)$. The shear modulus and the Poisson's ratio of the half-space are denoted by $G^{(2)}$ and $\nu^{(2)}$, respectively. Our aim is to find the stresses and strains in the inhomogeneous layer.

Validity of the operations that follow is ensured by assuming that the functions $G^{(1)}$ and $\nu^{(1)}$ are bounded and Riemann-integrable in the interval $[0, h]$ and, that $Z(r, \beta, z)$ can be written as a double Fourier series in β and a Hankel integral in r

$$Z(r, \beta, z) = \sum_{m=-\infty}^{+\infty} e^{im\beta}\int_0^\infty g(z, \alpha, m) J_m(\alpha r)\,\alpha d\alpha \tag{2.1}$$

$$g(z, \alpha, m) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty Z(r, \beta, z) J_m(\alpha r)\, r e^{-im\beta} dr d\beta$$

where $J_m(\alpha r)$ is a m-th order Bessel function of the first kind.

The boundary conditions of the problem at the layer surface $(z = 0)$ have the form

$$\sigma_z^{(1)} = \tau_{rz}^{(1)} = \tau_{\beta z}^{(1)} = 0 \tag{2.2}$$

Here and in the following the superscript 1 refers to the layer and the superscript 2 to the half-space.

At the plane of contact between the layer and the half-space $(z = h)$ the conditions of perfect bond are the boundary conditions. This means that the following equations must hold:

$$u_r^{(1)} = u_r^{(2)}, \quad u_\beta^{(1)} = u_\beta^{(2)}, \quad u_z^{(1)} = u_z^{(2)}, \tag{2.3}$$

$$\sigma_z^{(1)} = \sigma_z^{(2)}, \quad \tau_{rz}^{(1)} = \tau_{rz}^{(2)}, \quad \tau_{\beta z}^{(1)} = \tau_{\beta z}^{(2)}$$

From the condition of the problem it follows that $\varphi_1 = \varphi_2 = 0$. Let us also set $N_1 = N_2 = 0$.

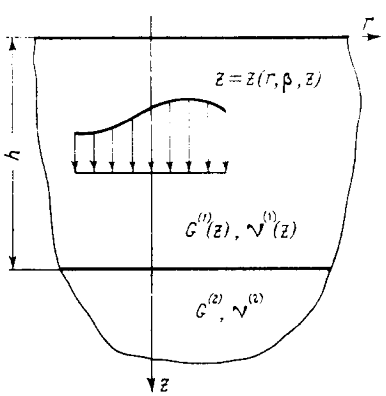

Fig. 1

The solution of the problem therefore consists of finding such solutions of (1.3) for the half-space and the layer, which would satisfy the boundary conditions (2.2) and (2.3).

We seek the solution of (1.3) in the form

$$\begin{Vmatrix} S_1^{(n)} \\ S_2^{(n)} \\ S_3^{(n)} \\ S_4^{(n)} \end{Vmatrix} = \sum_{m=-\infty}^{+\infty} e^{im\beta} \int_0^\infty J_m(\alpha r) \begin{Vmatrix} \frac{1}{\alpha} s_1^{(n)}(z, \alpha, m) \\ s_2^{(n)}(z, \alpha, m) \\ \alpha s_3^{(n)}(z, \alpha, m) \\ s_4^{(n)}(z, \alpha, m) \end{Vmatrix} d\alpha \quad (n = 1, 2) \qquad (2.4)$$

where $s_j^{(n)}(z, \alpha, m)$ $(j = 1, 2, 3, 4)$ are functions to be determined. Substituting $S_j^{(n)}$ from (2.4) and Z from (2.1) into Eqs. (1.3), we obtain a system of ordinary differential equations for the functions $s_j^{(n)}(z, \alpha, m)$ which in the matrix form become

$$dS^{(n)}/dz = \alpha A^{(n)} S^{(n)} - g^{(n)} D \quad (n = 1, 2) \qquad (2.5)$$

$$A^{(n)}(G^{(n)}, \nu^{(n)}) = \begin{Vmatrix} 0 & -1 & 0 & \frac{1}{G^{(n)}} \\ \frac{\nu^{(n)}}{1-\nu^{(n)}} & 0 & \frac{1-2\nu^{(n)}}{2(1-\nu^{(n)})G^{(n)}} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{2G^{(n)}}{1-\nu^{(n)}} & 0 & -\frac{\nu^{(n)}}{1-\nu^{(n)}} & 0 \end{Vmatrix}$$

$$S^{(n)} = \begin{Vmatrix} s_1^{(n)}(z, \alpha, m) \\ \cdot \\ \cdot \\ s_4^{(n)}(z, \alpha, m) \end{Vmatrix}, \quad D = \begin{Vmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Vmatrix}, \quad \begin{matrix} g^{(1)} = g(z, \alpha, m) \\ g^{(2)} = 0 \end{matrix}$$

Using the relations (1.6) to determine the components of the stress tensor and displacement vector and substituting the resulting expressions into the boundary conditions (2.2) and (2.3), we obtain

$$s_3^{(1)}(0, \alpha, m) = s_4^{(1)}(0, \alpha, m) = 0 \qquad (2.6)$$

$$s_j^{(1)}(h, \alpha, m) = s_j^{(2)}(h, \alpha, m) \quad (j = 1, 2, 3, 4)$$

Moreover, from the condition of the boundedness of the stresses and displacements at infinity it follows, that the functions $s_j^{(2)}(z, \alpha, m)$ must be bounded when $z \to \infty$. Thus the three-dimensional problem of the theory of elasticity is reduced to the boundary value problems for the ordinary linear differential equations (2.5).

Let us denote by $\Omega_0^z(\alpha A^{(1)})$ the matrizant of the homogeneous system corresponding to the system (2.5) with $n = 1$. Then the solution of the inhomogeneous system with the boundary conditions $S^{(1)}|_{z=0} = S_0$ is written in the form

$$S^{(1)} = \Omega_0^z(\alpha A^{(1)}) S_0 - KD \qquad (2.7)$$

$$K = \int_0^z \Omega_t^z(\alpha A^{(1)}) g(t, \alpha, m)\, dt$$

The solution simplifies considerably when the mass forces can be expressed in the form

$$Z = \delta(z - h_1) f(r, \beta) \tag{2.8}$$

$$f(r, \beta) = \sum_{m=-\infty}^{+\infty} e^{im\beta} \int_0^\infty g(\alpha, m) J_m(\alpha r) \alpha d\alpha$$

where ($\delta(z - h_1)$ is the Dirac delta function. This is the case when the mass forces represent the load $f(r, \beta)$ applied within the layer at the level $z = h_1$ and acting in the direction of the z-axis. Then, replacing in (2.7) $g(t, \alpha, m)$ by $\delta(t - h_1)\, g(\alpha, m)$ and integrating, we obtain

$$K = \begin{cases} 0, & z < h_1 \\ \Omega_{h_1}^z(\alpha A^{(1)})\, g(\alpha, m), & z \geqslant h_1 \end{cases} \tag{2.9}$$

When the load is applied at the surface of the inhomogeneous layer, we must set $h_1 = 0$ in (2.9).

When the matrizant is known, the computation of the components of the displacement vector and stress tensor obviously requires the knowledge of all elements of the column matrix S_0. The elements of the third and fourth row are, according to the conditions (2.6), equal to zero. The remaining two rows, $s_1^{(1)}(0, \alpha, m)$ and $s_2^{(1)}(0, \alpha, m)$ can be found from the boundary conditions at $z = h$. To do this we need the solution of (2.5) for the half-space ($n = 2$). We assume that the solution bounded at infinity has the form (2.10)

$$s_1^{(2)} = -\frac{1}{2G^{(2)}}[(1 - 2\nu^{(2)}) C_1 + 2(1 - \nu^{(2)}) C_2 - \alpha(z - h)(C_1 + C_2)] e^{-\alpha(z-h)}$$

$$s_2^{(2)} = -\frac{1}{2G^{(2)}}[2(1 - \nu^{(2)}) C_1 + (1 - 2\nu^{(2)}) C_2 + \alpha(z - h)(C_1 + C_2)] e^{-\alpha(z-h)}$$

$$s_3^{(2)} = [C_1 + \alpha(z - h)(C_1 + C_2)] e^{-\alpha(z-h)}$$

$$s_4^{(2)} = [C_2 - \alpha(z - h)(C_1 + C_2)] e^{-\alpha(z-h)}$$

Here and in the following $C_k (k = 1, 2, \ldots)$ are functions of m and of the parameter α and can be determined from the boundary conditions (2.6). Let us introduce the following notation:

$$S^{(1)}|_{z=h} = \left\| \begin{matrix} s_1^{(1)}(h, \alpha, m) \\ s_2^{(1)}(h, \alpha, m) \\ \hdashline s_3^{(1)}(h, \alpha, m) \\ s_4^{(1)}(h, \alpha, m) \end{matrix} \right\| = \left\| \begin{matrix} U_1(h) \\ \hdashline U_2(h) \end{matrix} \right\| \tag{2.11}$$

$$S_0 = \left\| \begin{matrix} s_1^{(1)}(0, \alpha, m) \\ s_2^{(1)}(0, \alpha, m) \\ \hdashline 0 \\ 0 \end{matrix} \right\| = \left\| \begin{matrix} U_1(0) \\ \hdashline 0 \end{matrix} \right\|$$

$$KD|_{z=h} = \left\| \begin{matrix} k_1(h, \alpha, m) \\ k_2(h, \alpha, m) \\ \hdashline k_3(h, \alpha, m) \\ k_4(h, \alpha, m) \end{matrix} \right\| = \left\| \begin{matrix} K_1 \\ \hdashline K_2 \end{matrix} \right\|, \quad \Omega_0^h(\alpha A^{(1)}) = \left\| \begin{matrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{matrix} \right\|$$

Here Ω_{kl} $(k = 1, 2;\ l = 1,2)$ are 2×2 sub-matrices of the matrix $\Omega_0^h(\alpha A^{(1)})$.

Setting $z = h$ in the formulas (2.7) and (2.10) we find from the boundary conditions (2.6) all unknown functions of the parameters m and α

$$U_1(0) = (\Omega_{11} + M\Omega_{21})^{-1}(K_1 + MK_2) \tag{2.12}$$

$$C = \Omega_{21}U_1(0) - K_2$$

$$M = \frac{1}{2G^{(2)}}\begin{Vmatrix} 1-2\nu^{(2)} & 2(1-\nu^{(2)}) \\ 2(1-\nu^{(2)}) & 1-2\nu^{(2)} \end{Vmatrix}, \quad C = \begin{Vmatrix} C_1 \\ C_2 \end{Vmatrix}$$

Thus the problem of determining the stress-strain state of the inhomogeneous layer and the half-space reduces, in the end, to computing improper integrals. Formulas for determining the displacements in the inhomogeneous layer and in the half-space follow:

$$u_r^{(n)} = -\sum_{m=-\infty}^{+\infty} e^{im\beta}\int_0^\infty \left[J_{m+1}(\alpha r) - \frac{m}{\alpha r}J_m(\alpha r)\right] s_1^{(n)}(z, \alpha, m)\, d\alpha \tag{2.13}$$

$$u_\beta^{(n)} = i\sum_{m=-\infty}^{+\infty} me^{im\beta}\int_0^\infty \frac{1}{\alpha r}J_m(\alpha r)\, s_1^{(n)}(z, \alpha, m)\, d\alpha$$

$$u_z^{(n)} = \sum_{m=-\infty}^{+\infty} e^{im\beta}\int_0^\infty J_m(\alpha r)\, s_2^{(n)}(z, \alpha, m)\, d\alpha \qquad (n = 1, 2)$$

If the mass forces Z are independent of the angular coordinate β, then the summation sign must be deleted from all the above formulas, since $g(z,\ \alpha,\ m) = 0$ for all $m \neq 0$ and the series are reduced to single terms each of which corresponds to the case $m = 0$.

3. Let us now turn our attention to the problem of finding the matrizant of the homogeneous system of differential equations corresponding to the system (2.5) with $n = 1$. If the shear modulus and the Poisson's ratio of the layer are constant, the elements of the matrix $A^{(1)}$ are also constant. The matrizant which reduces to E (E is the unit matrix) when $z = z_0$, has the form [8]

$$\Omega_{z_0}^{\ z}(\alpha A^{(1)}) = e^{A^{(1)}\alpha(z-z_0)} \tag{3.1}$$

The elements of this matrix can be computed without difficulty. Substituting the expression (3.1) into the formulas of Sect. 2, we arrive at the problem of equilibrium of a homogeneous layer bonded with a half-space and acted upon by the mass forces Z. A particular case of this problem was dealt with in [9], where the authors investigated the stress-strain state of an elastic layer resting on a rigid support and acted upon by a concentrated force P within the layer. The formulas of this paper can be obtained from our formulas given above, by making additional assumptions, namely that $G^{(2)} \to \infty$ and, that in the expression (2.9)

$$g(\alpha,\ m) = \begin{cases} P/(2\pi), & m = 0 \\ 0, & m \neq 0 \end{cases}$$

The matrizant of the homogeneous system of differential equations corresponding to (2.5) is expressed by a matrix exponent also in the case when the shear modulus of the layer varies with depth according to an exponential law of the form

$$G^{(1)}(z) = G_0 e^{2bz}$$

with a constant Poisson's ratio.

To prove this statement, we shall introduce a new unknown T using the following transformation

$$S^{(1)} = FT, \qquad F(z, b) = \left\| \begin{matrix} e^{-bz} & 0 & 0 & 0 \\ 0 & e^{-bz} & 0 & 0 \\ 0 & 0 & e^{bz} & 0 \\ 0 & 0 & 0 & e^{bz} \end{matrix} \right\| \tag{3.2}$$

As the result, we arrive at the following matrix equation:

$$dT/dz = BT, \qquad B = F^{-1}\left(\alpha A^{(1)} F - \frac{dF}{dz}\right) \tag{3.3}$$

$$B(\alpha, b, G_0, \nu^{(1)}) = \left\| \begin{matrix} b & -\alpha & 0 & \dfrac{\alpha}{G_0} \\ \dfrac{\alpha\nu^{(1)}}{1-\nu^{(1)}} & b & \dfrac{\alpha(1-2\nu^{(1)})}{2(1-\nu^{(1)})G_0} & 0 \\ 0 & 0 & -b & \alpha \\ \dfrac{2\alpha G_0}{1-\nu^{(1)}} & 0 & -\dfrac{\alpha\nu^{(1)}}{1-\nu^{(1)}} & -b \end{matrix} \right\|$$

The elements of the matrix B are constant, therefore we have

$$\Omega_{z_0}^{z}(\alpha A^{(1)}) = F e^{B(z-z_0)} F^*, \qquad F^* = F^{-1}|_{z=z_0} \tag{3.4}$$

Let us now assume that the interval $[0, h]$ of variation of the variable z can be subdivided by means of the points $z_1, z_2, \ldots$ into segments, within which the shear modulus is either constant, or varies exponentially, and the Poisson's ratio $\nu^{(1)} = \text{const}$. In this case, to obtain the solution of the homogeneous system of differential equations we must utilize the following property of the matrizant [8]:

$$\Omega_{z_0}^{z}(\alpha A^{(1)}) = \Omega_{z_{n-1}}^{z}(\alpha A^{(1)})\,\Omega_{z_{n-2}}^{z_{n-1}}(\alpha A^{(1)}) \ldots \Omega_{z_0}^{z_1}(\alpha A^{(1)}) \tag{3.5}$$

$$(0 \leqslant z_0 \leqslant z_1 \leqslant \ldots \leqslant z \in [0, h])$$

In accordance with the argument given above, the matrices $\Omega_{z_{k-1}}^{z_k}(\alpha A^{(1)})$ $(k = 1, 2, \ldots, n;\ z_n = z)$, appearing in (3.5) can be represented by the exponential matrix functions.

We illustrate the results obtained by considering a problem which arises in the analysis of renewable road surfaces. An inhomogeneous layer the elastic characteristics of which vary with depth according to the relation depicted in Fig. 2, is bonded in a continuous manner to a homogeneous elastic half-space and is acted upon by forces q normal to the surface and uniformly distributed over a circular area of radius δ. We must determine the displacement $u_z^{(1)}$ of the point lying at the coordinate origin.

The matrizant of a homogeneous system corresponding to (2.5) has, in accordance with the expressions (3.1), (3.4) and (3.5), the form

$$\Omega_0^{\,z}(\alpha A^{(1)}) = \begin{cases} V(z), & 0 \leqslant z \leqslant z_1 \\ \Omega_{z_1}^{\,z}(\alpha A^{(1)})\, V(z_1), & z_1 < z \leqslant z_2 \\ \Omega_{z_2}^{\,z}(\alpha A^{(1)})\, \Omega_{z_1}^{\,z_2}(\alpha A^{(1)})\, V(z_1), & z_2 < z \leqslant h \end{cases} \tag{3.6}$$

$$V(z) = F(z,\ b_1)e^{B_1 z}, \qquad B_1 = B(\alpha,\ b_1,\ G_0,\ \nu_1)$$

$$\Omega_{z_1}^{\,z}(\alpha A^{(1)}) = e^{A\alpha(z-z_1)}, \qquad A = A^{(1)}(G_2,\ \nu_2)$$

$$\Omega_{z_2}^{\,z}(\alpha A^{(1)}) = F(z - z_2,\ b_2)e^{B_2(z-z_2)}, \qquad B_2 = B(\alpha,\ b_2,\ G_3,\ \nu_3)$$

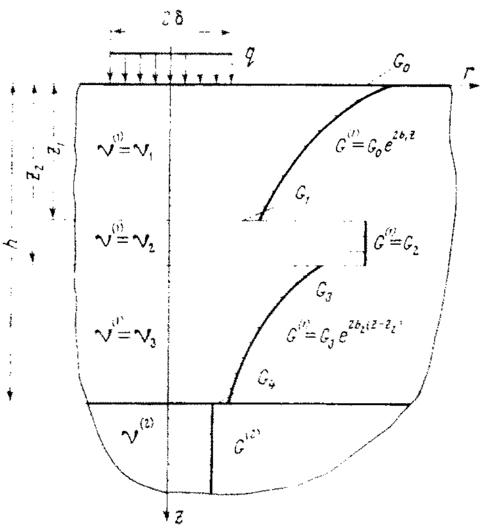

Fig. 2

Using the matrizant (3.6) we find all elements of the matrices S_0 and $S^{(1)}$ from the formulas (2.12) and (2.7), and after this the problem reduces to computing an improper integral for the component $u_z^{(1)}$ for $z = r = 0$.

Some of the results obtained on the digital computer are given in Table 1. The computations were performed for the case $G^{(2)} = G_4$ and $\nu_1 = \nu_2 = \nu_3 = \nu^{(2)} = {}^1/_3$. The penultimate column gives the values of the dimensionless quantity $W = u_z^{(1)}G_0h/P$, where $P = \pi\delta^2 q$ is the resultant of the external load. The last column gives the quantity R which characterizes the deformability of the elastic system in question. It shows how much smaller is the maximum sag of a homogeneous half-space with the modulus G_0 under a given load, compared with that of the inhomogeneous body in question under the same load.

When the shear modulus varies with depth according to a relation of the form

$$G^{(1)} = G_0(1 + cz)^b \tag{3.7}$$

Table 1

$\frac{G_0}{G_1}$	$\frac{G_2}{G_1}$	$\frac{G_2}{G_3}$	$\frac{G_3}{G_4}$	$\frac{z_1}{\delta}$	$\frac{z_2 - z_1}{\delta}$	$\frac{h - z_2}{\delta}$	W	R
4	3	2	4	3	1	2	3.16	2.48
4	3	2	4	1	1	2	2.78	3.27
4	3	2	4	3	$^3/_2$	4	4.13	2.29
4	3	4	8	1	1	2	5.87	6.92
4	3	4	8	1	$^1/_2$	2	6.09	8.20
4	3	4	8	2	1	3	6.60	5.18
4	3	2	8	1	1	2	3.83	4.51
2	$^1/_3$	1	2	$^1/_3$	0	1	2.23	7.87
2	$^1/_3$	1	4	$^1/_3$	0	1	3.59	12.7
2	$^1/_3$	1	$^1/_3$	$^1/_3$	$^1/_2$	1	1.30	3.35

and the Poisson's ratio $\nu^{(1)} = \text{const}$. the matrizant elements can be written in terms of special functions. To prove it we introduce a function ψ such that

$$s_1^{(1)} = \frac{\alpha}{2G^{(1)}}\left[(1-\nu^{(1)})\frac{d^2}{dz^2} + \alpha^2\nu^{(1)}\right]\psi, \qquad s_3^{(1)} = \alpha^3\psi \tag{3.8}$$

$$s_2^{(1)} = \left[\frac{\alpha^2}{G^{(1)}}\frac{d}{dz} - \frac{d}{dz}\left(\frac{1-\nu^{(1)}}{2G^{(1)}}\frac{d^2}{dz^2} + \frac{\alpha^2\nu^{(1)}}{2G^{(1)}}\right)\right]\psi, \quad s_4^{(1)} = \alpha^2\frac{d\psi}{dz}$$

Substituting the expressions (3.8) into the homogeneous system of differential equations corresponding to (2.5), we obtain the following equation for ψ :

$$\left(\frac{d^2}{dz^2} - \alpha^2\right)\left[\frac{1-\nu^{(1)}}{G^{(1)}}\left(\frac{d^2}{dz^2} - \alpha^2\right)\psi\right] + \alpha^2\psi\frac{d^2}{dz^2}\left(\frac{1}{G^{(1)}}\right) = 0 \tag{3.9}$$

A general solution of this equation for the shear modulus of the form (3.7) was given in [7, 10]. Using this solution we can easily find from (3.8) the functions $s_j^{(1)}$ ($j = 1, 2, 3, 4$) and thus construct the fundamental matrix of the system (2.5) for $g^{(1)} = 0$. The four arbitrary constants should be arranged in such a way that the fundamental matrix becomes a unit matrix when $z = z_0$. This matrix is, by definition, the matrizant.

We can easily obtain the solution of (3.9) in the case when the shear modulus varies according to the law $G_{(z)}^{(1)} = G_0 / (1 + cz)$, and the Poisson's ratio is an arbitrary function of z. The solution has the form

$$\psi = C_1e^{\alpha z} + C_2e^{-\alpha z} + C_3\left[e^{\alpha z}\int_0^z \chi(t)\,dt - e^{-\alpha z}\int_0^z \chi(t)\,e^{2\alpha t}dt\right] +$$

$$C_4\left[e^{-\alpha z}\int_0^z \chi(t)\,dt - e^{\alpha z}\int_0^z \chi(t)\,e^{-2\alpha t}dt\right], \quad \chi(z) = G^{(1)}(z) / [1 - \nu^{(1)}(z)]$$

Let us now consider the most general case of variation in the elastic properties in which the shear modulus and the Poisson's ratio of the layer are bounded and Riemann-integrable functions of the coordinate z. In this case the matrizant $\Omega_{z_0}{}^z(\alpha A^{(1)})$ can be found using the following representation of the operator [8]:

$$\Omega_{z_0}{}^z(\ldots) = E + \int_{z_0}^z(\ldots)\,dz + \int_{z_0}^z(\ldots)\,dz\int_{z_0}^z(\ldots)\,dz + \cdots \tag{3.10}$$

Applying this operator to the matrix $\alpha A^{(1)}$ and taking a sufficient number of terms, we can reach any prescribed degree of accuracy. In most cases however, it is simpler to employ the expression for the matrizant in the form of the following multiplicative integral [8]:

$$\Omega_{z_0}{}^z(\alpha A^{(1)}) = \widehat{\int}_{z_0}^{z}(E + \alpha A^{(1)}dz) \tag{3.11}$$

To compute this integral, we divide the interval $[z_0, z]$ into j arbitrary segments introducing the intermediate points $z_1, z_2, \ldots, z_{j-1}$ and setting $\Delta^* z_k = z_k - z_{k-1}$ ($k = 1, 2, \ldots, j$; $z_j = z$). We select the point ζ_k ($k = 1, 2, \ldots, j$) from the interval $[z_{k-1}, z_k]$ and denote

$$A^{(1)}\big|_{z=\zeta_k} = A_k$$

Since the multiplicative integral is defined by the following integral products [8]:

$$\int_{z_0}^{z} (E + \alpha A^{(1)}dz) = \lim_{\substack{\Delta^* z_k \to 0 \\ j\to\infty}} [e^{\alpha A_j \Delta^* z_k} \ldots e^{\alpha A_2 \Delta^* z_2} e^{\alpha A_1 \Delta^* z_1}] \tag{3.12}$$

$$\int_{z_0}^{z} (E + \alpha A^{(1)}dz) = \lim_{\substack{\Delta^* z_k \to 0 \\ j\to\infty}} (E + \alpha A_j \Delta^* z_j) \ldots (E + \alpha A_1 \Delta^* z_1) \tag{3.13}$$

we can use any of these expressions to compute the approximate value of the matrizant, by assigning to j some finite values. The formulas (3.12) and (3.13) are particularly suitable for digital machine computation.

As we said before, a particular case of the problem under consideration was dealt with in [5, 6] where the sag of an inhomogeneous layer resting on a rigid support, caused by the action of a normal axisymmetric load applied at the boundary surface was studied. It was assumed that the Poisson's ratio was constant and the shear modulus $E(z)$ varied continuously with depth. The final formula for computing the sags was represented by an improper integral of a function, the determination of which required the computation of a multiplicative integral formally resembling (3.13). However, comparing the matrix $A^{(1)}$ with the corresponding matrix given in [5, 6], we come to the conclusion that the elements of the latter matrix are more complicated and less suitable for computations, and contain the derivatives $dE(z) / dz$. This implies that the solution of the problem in question given in the present paper is simpler, and therefore preferable.

4. Let us show how an analogous problem in which the mass forces X act on the layer, should be solved. A particular case of this problem in which a shearing force acts at the boundary surface, is of interest in the problem of computing the action of forces on road coverings. In solving the problem we must assume that the function $X = X(r, \beta, z)$ can be represented by a double Fourier series in β and a Hankel integral in r, i.e. in the form of (2.1) in which Z is replaced by X. Then the formulas (1.2) yield

$$\varphi_1 = -\partial\Psi / \partial x, \qquad \varphi_2 = -\partial\Psi / \partial y$$

$$\Psi = \sum_{m=-\infty}^{+\infty} e^{im\beta} \int_0^\infty g(z, \alpha, m) J_m(\alpha r) \frac{d\alpha}{\alpha}$$

The form in which the functions φ_1 and φ_2 appear, suggests the form which the unknown functions S_j $(j = 1, 2, 3, 4)$, N_1 and N_2 must assume to make the variables in (1.3) and (1.4) separable. This will make it possible to reduce the initial three-dimensional problem of the theory of elasticity to the boundary value problems for ordinary differential equations. The remaining part of the procedure is similar to that already given.

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DEFORMATION OF AN ELASTIC WEDGE REINFORCED BY A BEAM

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Two problems of plane strain of an elastic infinite wedge reinforced by an infinite constant-thickness beam are considered. In the first problem the beam is welded to the wedge along the bissectrix and is in complete contact with it. A longitudinal force, a transverse force, and a bending moment are applied to the end of the beam and arbitrary normal and tangential stresses are given on the boundary surfaces of the wedge. In the second problem, the beam is in contact without friction with one face of the wedge, arbitrary stress resultants act on both the wedge and the beam. Both problems are reduced to first-order difference equations and are solved in closed form.

1. In an elastic wedge let $0 \leqslant r < \infty$, $-\alpha \leqslant \theta \leqslant \alpha$, an elastic beam $2h$ thick (Fig. 1) is welded along the $\theta = 0$ axis, and the contact surfaces of the wedge and beam are connected completely. A longitudinal tensile force $2T$, a bending moment $2M$, a transverse force $2P$ or another load causing equivalent stress resultants at the point $r=0$ of the beam act on the free part of the beam $\theta = \pi$. Concentrated forces, a normal $2N$ and tangential $2S$, are applied to the wedge at its face $\theta = \alpha$ as an arbitrary load.